Sec 1.3

Exponential Functions

Goals

- To determine the domain, range, and graph of an exponential function.
- To solve problems involving exponential growth and decay.
- To use exponential regression equations to solve problems.

The Savings Bond…

- Mr. Maierhofer was given a $75 savings bond in 1965. It was originally purchased for $37.50 and was to be worth the $75 after about 10 years. He didn't cash it in, and he found out in the summer of 2006 that it was fully matured (no longer earning interest). If the interest rate of the bond was 6.89%, how much was the bond worth when he cashed it in last summer?

Exponential Growth

- This problem is an example of an exponential growth problem
- Compound interest problems are modeled by the function
  - $y = P \cdot a^x$
  - P = the initial investment amount
  - a = (1 + the annual interest rate (as a decimal))
  - x is the number of years

The Savings Bond…

So what was Mr. Maierhofer's savings bond worth?

- Originally purchased in the summer of 1965 for $37.50
- Earned 6.89% interest
- Was cashed in the summer of 2005
- $y = P \cdot a^x$

Exponential Function

Definition of Exponential Function

- Let a be a positive real number other than 1. The function:
  - $f(x) = a^x$
- is the **exponential function** with base *a*

Rules for Exponents

$$a^x \cdot a^y = a^{x+y}$$

$$\frac{a^x}{a^y} = a^{x-y}$$

$$(a^x)^y = (a^y)^x = a^{xy}$$

$$a^x \cdot b^x = (ab)^x$$

$$\left(\frac{a}{b}\right)^x = \frac{a^x}{b^x}$$

If $a>0$ and $b>0$, the following hold for all real numbers x and y.

Population Growth

Table 1.7 World Population

Year	Population (millions)	Ratio	
1986	4936		
1987	5023	$5023/4936 \approx 1.0176$	1.76%
1988	5111	$5111/5023 \approx 1.0175$	1.75%
1989	5201	$5201/5111 \approx 1.0176$	1.76%
1990	5329	$5329/5201 \approx 1.0246$	~~2.46%~~
1991	5422	$5422/5329 \approx 1.0175$	1.75%

Source: Statistical Office of the United Nations, *Monthly Bulletin of Statistics*, 1991.

By about what percent is the population increasing annually?

Population Growth

- Population growth can also be modeled by an exponential function.
- The population of the world was 4936 million in 1986. It is increasing by about 1.8% per year. What will the population be in 2010?

$$y = 4936(1+.018)^{24}$$

$y = 7574$ million

Exponential Decay

- Exponential functions can also be used to model a decrease over time, such as radioactive decay.
- **Half-life**: the amount of time it takes for half of a radioactive substance to change from its radioactive state to its nonradioactive state.

Modeling Radioactive Decay

- Suppose the half-life of a certain radioactive substance is 20 days and that there are 5 grams present initially. When will there be only 1 gram of the substance remaining?
- How much after 20 days? 2.5
- After 40 days? 1.25

60 ? 0.625

Modeling Radioactive Decay

- After *t* days?

$y = p\,a^{t}$ $\quad p = \text{initial } (5) \quad a = \frac{1}{2}$

$y = 5\left(\frac{1}{2}\right)^{t/20}$ $\quad t = 20 \quad y = 5\left(\frac{1}{2}\right)^{1} = 2.5$

$t = 40 \quad y = 5\left(\frac{1}{2}\right)^{2} = \left(5 \cdot \frac{1}{2}\right)\frac{1}{2} = 1.25$

- When will there be only 1 gram of the substance remaining? Use a graphing calculator to solve it graphically.

$1 = 5\left(\frac{1}{2}\right)^{t/20}$

$t = 46.44$ days

Exponential Growth, Exponential Decay

- Compound interest investments, population growth, and radioactive decay are all examples of *exponential growth and decay*.
- The function $y = k \cdot a^x$, $k > 0$ is a model for **exponential growth** if a > 1.

  1+ %   1.0689   1.018

- It is a model for **exponential decay** if

  $0 < a < 1$

Example 3

- Predicting the Population: Use the population data table to estimate the population for the year 1990. Compare it with the actual population of approximately 250 million in 1990.

$y = 203.3(1+.170)^{t=2}$

278.3 million

Table 1.8 U.S. Population

Year	Population (millions)	
1880	50.2	
1890	63.0	.255
1900	76.0	.211
1910	92.0	.211
1920	105.7	.149
1930	122.8	.161
1940	131.7	.072
1950	151.3	.149
1960	179.3	.185
1970	203.3	.134

Source: The Statesman's Yearbook, 129th ed. (London: The Macmillan Press, Ltd., 1992).

.170

Example 3

- Let x=0 be 1880, x=1 is 1890, etc.
- Enter the years into L1 and the population into L2 (STAT, 1 Edit)
- Set up your window
- Turn on stat plot (2nd Y=)
- Do an exponential regression (STAT, CALC, 0 ExpReg)
- Store that exp reg in y1 (VARS, 5 Stats, EQ, 1 RegEQ)
- Graph
- How could we use the graph to find the population in 1990?

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1960	179.3
1970	203.3

Source: The Statesman's Yearbook, 129th ed. (London: The Macmillan Press, Ltd., 1992).

The number *e*

- *e* is defined as the value the function $f(x)=(1+1/x)^x$ approaches as *x* approaches infinity $\lim_{x\to\infty}(1+\frac{1}{x})^x = e$
- Plug this equation in your calculator and look at the table as *x* gets "big".
- Look at the graph as *x* gets "big".
- $e = 2.718281828459045235360287471 35$

The number *e*

- The exponential functions $y = e^x$ and $y = e^{-x}$ are often used as models of exponential growth or decay.

Compound Interest

- Interest can be compounded annually, monthly, daily, or even continuously.
- If interest is compounded continuously, we use the model $y = P \cdot e^{rt}$
- P = initial investment
- r = interest rate
- t = time in years.

Example $y = Pe^{rt}$

- If Harmony invests \$2500 in a savings account with a 7% interest rate compounded annually, how long will it take until Harmony's account has a balance of \$5000?

$$\frac{5000}{2500} = \frac{2500(1+.07)^t}{2500}$$

$$2 = (1+.07)^t$$

$$t \approx 10.24 \text{ yrs.}$$

Example Cont'd

- What if it is compounded monthly?

$2 = \left(1 + \frac{.07}{12}\right)^t \approx 10$ yrs.

Example Cont'd

- What if it is compounded daily?

$$2 = \left(1 + \frac{.07}{365}\right)^{365t} \approx 9.9 \text{ yrs.}$$

Example Cont'd

- What if it is compounded continuously?

$y = Pe^{rt}$

$\frac{5000}{2500} = \frac{2500 e^{.07t}}{2500}$

$2 = e^{.07t}$

Assignment

- Sec 1.3, p. 24 -26: 1-6, 7, 11-16, 23-29 odd, 30-38 even